The Hartogs-type extension theorem for meromorphic mappings into q-complete complex spaces

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0. Introduction.

The aim of this note is to prove a result on extension of meromorphic mappings, which can be considered as a direct generalisation of the Hartogs extension theorem for holomorphic functions.

Let Δ_r^n be the polydisc of radius r in $\mathbf{C^n}$, and set $\Delta^n := \Delta_1^n$. Let us define the "q-concave" Hartogs figure $H_n^q(r)$ as the following open set in $\mathbf{C^{n+q}}$:

$$H_n^q(r) := \Delta^n \times (\Delta^q \setminus \bar{\Delta}_{1-r}^q) \cup \Delta_r^n \times \Delta^q. \tag{1}$$

Note that $H_n^q(r)$ has Δ^{n+q} as its envelope of holomorphy.

Let Y be a reduced complex space. Meromorphic mappings with values in Y are sad to satisfy a Hartogs-type extension Theorem if any such $f : H_n^q(r) \longrightarrow Y$ extends to a meromorphic map map $\hat{f} : \Delta^{n+q} \to Y$ from the unit polydisk Δ^{n+q} into Y. Sometimes we shall say more precisely that meromorphic maps into such Y possed a meromorphic extension property in bidimension (n,q).

Hartogs-type extension Theorems for meromorphic mappings have been proved when Y is compact Kähler, and when Y has some weaker metric properties by the first author in [I$_1$], [I$_2$] and [I$_3$]. In this note we shall prove the following

Theorem. *Every meromorphic mapping $f : H_n^q(r) \longrightarrow Y$, where Y is a q-complete complex space, extends to a meromorphic mapping from Δ^{n+q} to Y.*

We recall that a *strictly q-convex* function ρ on the complex space Y with $dimY = N$ is a real valued C^2 function such that the hermitian matrix consisting of the coefficients of the $(1,1)$ real form $dd^c\rho$ has at least $N-q+1$ positive eigenvalues at all points of Y. (Smooth objects on a complex space Y are by definition the pull-backs of smooth objects in domains of $\mathbf{C^M}$ under appropriate local embeddings. The number q is independent of such embeddings).

The complex space Y is called *q-complete* if there exists a strictly q-convex exhaustion function $\rho : Y \longrightarrow R^+$.

We remark that in the case $q=1$, i.e. when Y is Stein, the statement of the Theorem, via proper embedding of Y into $\mathbf{C^M}$, reduces to the extension of holomorphic functions. This is given by the classical theorem of Hartogs, [H].

More generally our Theorem provides Hartogs' type extension of meromorphic mappings into a complex subspaces of $\mathbf{CP^N} \setminus \mathbf{CP^{N-q}}$, see paragraph 3. Note that the Stein case inkludes here as $\mathbf{CP^N} \setminus \mathbf{CP^{N-1}}$.

One more point, which we would like to mention in the Introduction is that our Theorem improves the following result due to K. Stein:

Let D be a domain in $\mathbb{C}^{q+2}, q \geq 1$ and $K \subset\subset D$ a compact subset in D with connected complement. Let further Y be a normal complex space of dimension q. Then every holomorphic mapping $f : D \setminus K \to Y$ extends holomorphically onto D.

Note now that every noncompact irreducible complex space of dimension q is q-complete, see [O]. So we have the following immediate corollary from our Theorem:

Corollary 1. *Let Y be an irreducible non compact complex analytic space of dimension q. Every meromorphic mapping $f : H_n^q(r) \longrightarrow Y, n \geq 1$, extends to a meromorphic mapping from Δ^{n+q} to Y.*

If Y is compact and $f : H_n^q(r) \longrightarrow Y$ is not surjective, then delete from Y one point, which is not in the image of f, and call the resulting space Y'. Corollary 1 now applies to $f : H_n^q(r) \to Y'$ and f again extends onto the corresponding polydics.

In particular,

for any domain $D \subset \mathbb{C}^{q+1}$ and any compact $K \subset\subset D$ with connected complement, any nonsurjective meromorphic mapping $f : D \setminus K \to Y$ extends meromorphically onto D.

In our problem section we shall discuss among other open questions also some ones arising from the attempts to remove the condition on Y to be noncompact in the last statement.

Table of content

1. Preliminaries.

Let X and Y be reduced complex spaces with X normal. A *meromorphic mapping* $f : X \longrightarrow Y$ is defined as an irreducible, locally irreducible analytic subset $\Gamma_f \subset X \times Y$ (the graph of f), such that the restriction to Γ_f, $\pi \mid_{\Gamma_f} : \Gamma_f \longrightarrow X$, of the natural projection $\pi : X \times Y \longrightarrow X$ is proper, surjective and generically one to one, see [R]. The set $f[x] := \{y \in Y : (x, y) \in \Gamma_f\}$ is a compact subvariety in Y. The set of points $x \in X$ such that $dim f[x] \geq 1$ is analytic by the Remmert proper mapping theorem and has codimension at least two, because of the condition of irreducibility of Γ_f. This set is called the *fundamental set* of f or the *set of points of indeterminancy* of f and will be denoted by F. If X_1 is a normal subspace of X, $X_1 \not\subset F$, we denote by $f \mid_{X_1}$ the meromorphic mapping with a graph equal to the (unique!) irreducible component of $\Gamma_f \cap (X_1 \times Y)$, which projects *onto* X_1.

We shall list now some statements needed for the proof of our Theorem. First of all let us define the set

$$E_n^q(r) = (\Delta^{n-1} \times \Delta_r \times \Delta^q) \cup (\Delta^{n-1} \times \Delta \times A^q(1-r, 1) = \Delta^{n-1} \times H_1^q(r). \tag{2}$$

Here $A^q(1-r,1) := \{z \in \mathbb{C}^q : 1-r < \|z\| < 1\}$, $\|\cdot\|$ is a polydisk norm in $\mathbb{C}^q$. The following lemma for $q=1$ can be found in [I_4], Lemma 2.2.1. Proof for any $q \geq 1$ is the same.

Lemma 1. *If any meromorphic map $f : E^q_n(r) \to Y$ extends to a meromorphic map $\hat{f} : \Delta^{n+q} \to Y$ then the space Y possesses a meromorphic extension property in bidimension (n,q).*

We shall make use also from one result on *meromorphic families of analytic subsets* from [I_4].

Let S be a set, and $W \subset\subset \mathbb{C}^q$ an open subset. W is equipped with the usual Euklidean metric from $\mathbb{C}^q$. Y is again some complex space.

Definition. *(i) By a family of q-dimensional analytic subsets in complex space $X = W \times Y$ we shall understand an subset $\mathcal{F} \subset S \times W \times X$ such that, for every $s \in S$ the set $\mathcal{F}_s = \mathcal{F} \cap \{s\} \times W \times X$ is a graph of a meromorphic mapping of W into X.*

(ii) If the set S is equipped with topology and the space X is equipped with some Hermitian metric h we say that the family $\mathcal{F}$ is continuous at point $s_0 \in S$ if $\mathcal{H}-\lim_{s\to s_0} \mathcal{F}_s = \mathcal{F}_{s_0}$.

(iii) When S is a complex space itself call the family $\mathcal{F}$ meromorphic if the closure $\hat{\mathcal{F}}$ of the set $\mathcal{F}$ is an analytic subset of $S \times W \times X$.

Here by $\mathcal{H} - \lim_{s\to s_0} \mathcal{F}_s$ we denote the limit of closed subsets of $\mathcal{F}_s$ in the Hausdorff metric on $W \times X$. $\mathcal{F}$ is continuous if it is continuous at each point of S. If W_0 is open in W then the restriction $\mathcal{F}_{W_0}$ is naturally defined as $\mathcal{F} \cap (S \times W_0 \times X)$.

The statement about meromorphic families we need can be formulated as follows. For the standart notions and facts from pluripotential theory we refer to [Kl].

Consider a meromorphic mapping $f : V \times W_0 \longrightarrow X$ into a complex space X, where V is a domain in $\mathbb{C}^p$. Let S be some closed subset of V and $s_0 \in S$ some accumulation point of S. Suppose that for each $s \in S$ the restriction $f_s = f|_{\{s\}\times W_0}$ meromorphically extends onto $W \supset\supset W_0$. We suppose additionally that there is a compact $K \subset\subset X$ such that for all $s \in S$ $f_s(W) \subset K$.

Let ν_j denotes the minima of volumes of j-dimensional compact analytic subsets contained in our compact $K \subset X$. $\nu_j > 0$, see Lemma 2.3.1 from [I_4]. Fix some $W_0 \subset\subset W_1 \subset\subset W$ and put

$$\nu = \min\{vol(A_{q-j}) \cdot \nu_j : j = 1, \ldots, q\}, \tag{3}$$

where A_{q-j} are running over all $(q-j)$-dimensional analytic subsets of W, intersecting $\bar{W}_1$. Clearly $\nu > 0$. In the following *Lemma* the volumes of graphs over W are taken. More precisely, having an Euklidean metric form $w_e = dd^c\|z\|^2$ on $W \subset \mathbb{C}^q$ and Hermitian metric form w_h on X, we consider Γ_{f_s} for $s \in S$ as an analytic subsets of $W \times X$ and their volumes are

$$vol(\Gamma_{f_s}) = \int_{\Gamma_{f_s}} (p_1^* w_e + p_2^* w_h)^q = \int_W (w_e + (p_1)_* p_2^* w_h)^q, \tag{4}$$

where $p_1 : W \times X \to W$ and $p_2 : W \times X \to X$ are natural projections.

Lemma 2. *Suppose that there exists a neighbourhood $U \ni s_0$ in V such that, for all $s_1, s_2 \in S \cap U$*

$$|vol(\Gamma_{f_{s_1}}) - vol(\Gamma_{f_{s_2}})| < \nu/2. \tag{5}$$

If s_0 is a locally regular point of S then there exists a neighbourhood $V_1 \ni s_0$ in V such, that f meromorphically extends onto $V_1 \times W_1$.

Further, slightly modifying arguments from [I_3] we shall derive now the following version of so called Continuity principle.

Let $f : H_n^q(r) \longrightarrow Y$ be a given meromorphic mapping. Let $A_s^q(1-r,1) := \{s\} \times A^q(1-r,1)$ for $s \in \Delta^n$. We suppose that for s in some nonempty subset $S \subset \Delta^n$ the restriction $f_s := f \mid_{A_s^q(1-r,1)}$ is well defined and extends meromorphically to the polydisc Δ^q. .

Lemma 3. *Suppose that $f : H_1^q(r) \longrightarrow Y$ is meromorphic and:*

(i) there is a compact $K \subset\subset Y$ such that $f(\Delta^1 \times A^q(1-r,1)) \subset K$ and $f(\Delta_s^q) \subset K$ for all $s \in S$;

(ii) there is a constant $C_{<}0\infty$ such that $vol(\Gamma_{f_s}) \leq C_0$ for all $s \in S$.

Then:

1. Either there is a neighborhood $U \ni 0$ in Δ^n and a meromorphic extension of f onto $U \times \Delta^q$, or

2. 0 is an isolated point of S.

The volumes here are measured with respect to the Euclidean metric on $\mathbf{C^q}$ and some Hermitian metric h on Y. The condition of boundedness in (ii) clearly does not depend on the particular choice of h. We shall refer to this statement as to C.P. The condition $n = 1$ is important here, see Example 1 in [I_3]. We shall also discuss the related questions in our problem section.

To derive the proof of this statement from the reasonings in [I_3] we shall need some notions and results from the theory of cycle spaces (due to D.Barlet, see [B_2]) as they where adapted to our "noncompact" situation in [I_3]. For the english spelling of the Barlet terminology we send an interested reader to [Fj].

Recall that an analytic cycle of dimension q in complex space Y is a formal sum $Z = \sum_j n_j Z_j$, where $\{Z_j\}$ is a locally finite sequence of analytic subsets (allwayse of pure dimension q) and n_j are positive integers called multiplicities of Z_j. $|Z| := \bigcup_j Z_j$-support of Z. All complex spaces in this paper are reduced, normal and countable at infinity.

With a given meromorphic mapping $f : \Delta \times A^q(1-r,1) \to X$, satifying conditions of Lemma 3 we associate the following space of cycles. Fix some $0 < c < 1$.

Consider a set $\mathcal{C}_{f,C}^{'}$ of all analytic cycles Z in $Y := \Delta^{1+q} \times X$ of pure dimension q, such that:

(a) $Z \cap [\Delta \times A^q(1-r,1] \cap X] = \Gamma_{f_z} \cap \{z\} \times A^q(1-r,1] \times X$ for some $z \in \Delta(c)$. This means, in particular, that for this z mapping f_z extends meromorphically from $A_z^q(1-r,1)$ onto Δ_z^q.

(b) $vol(Z) < C$, where C is a some constant, $C > C_0$, C_0 beeing from Lemma 3.

Define $\bar{\mathcal{C}}_{f,C}$ to be a closure of $\mathcal{C}'_{f,C}$ in the usual topology of currents, see below. In [I_3] it was shawn that $\mathcal{C}_{f,C} := \{Z \in \bar{\mathcal{C}}_{f,C} : vol(Z) < C\}$ is an analytic space of finite dimension in the neighborhood of each of its points.

As we already had mentioned our first aime is to prove the analyticity of $\mathcal{C}_{f,C}$.

Let $f : \Delta \times A^q(1-r,1] \to X$ be our map. Denote by $\mathcal{C}_0$ the subset of $\bar{\mathcal{C}}_{f,C}$ consisting of cycles which are limits of $\{\Gamma_{f_{s_n}}\}$ for $s_n \to 0, s_n \in S$. This is a compact subset (by Bishop's theorem) of the topological space $\mathcal{C}_{f,2C}$. For every cycle $Z \in \mathcal{C}_0$ define its neighborhood W_Z as above. Let $W_{Z_1}, ..., W_{Z_N}$ be a finite covering of $\mathcal{C}_0$. Remark that there is an $\varepsilon_0 > 0$ such that for any $s \in S \cap \Delta(\varepsilon_0)$ we have $\Gamma_{f_s} \subset \bigcup_{j=1}^N W_{Z_j}$.

Now we are prepared to sketch the proof of Lemma 3. Consider a universal family $\mathcal{Z} := \{Z_a : a \in \mathcal{C}_{f,2C_0}\}$. This is complex space of finite dimension . We have an evaluation map

$$F : \mathcal{Z} \to \Delta^{1+q} \times X$$

defined by $Z_a \in \mathcal{Z} \to Z_a \subset \Delta^{1+q} \times X$. Consider the union $\hat{\mathcal{C}}_0$ of those components of $\mathcal{C}_{f,2C_0}$ which intersect $\mathcal{C}_0$. Recall, that $\mathcal{C}_0$ stands here for the set of all limits of $\{\Gamma_{f_{s_n}}, s_n \in S\}$. At least one of those components, say $\mathcal{K}$, containes two points s_1 and s_2 s.t. Z_{s_1} projects onto Δ_0^k and Z_{s_2} projects onto Δ_s^k with $s \neq 0$. This is just because S contains more then one point. Consider the restriction $\mathcal{Z}\mid_{\mathcal{K}}$ of the universal space onto $\mathcal{K}$. This is an irreducible complex space of finite dimension. Take points $z_1 \in Z_{s_1}$ and $z_2 \in Z_{s_2}$ and join them by an analytic disk $\phi : \Delta \to \mathcal{Z}\mid_{\mathcal{K}}$, $\phi(0) = z_1, \phi(1/2) = z_2$. Then the composition $\psi = \pi \circ F \circ \phi : \Delta \to \Delta$ is not degenerate because $\psi(0) = 0 \neq s = \psi(1/2)$. Thus ψ is proper and obviously so is the map $F : \mathcal{Z}\mid_{\phi(\Delta)} \to F(\mathcal{Z}\mid_{\phi(\Delta)}) \subset \Delta^{1+q} \times X$. Thus $F(\mathcal{Z}\mid_{\phi(\Delta)})$ is an analytic set in $U \times \Delta^k \times X$ for small enough U extending Γ_f by the reason of dimension.

We also shall make use also of the following result due to D. Barlet, see [B_1] Proposition 3:

Lemma 4. *Let X be a reduced complex space (of finite dimension) and let $\rho : X \longrightarrow \mathbf{R}^+$ be a strictly q-convex function. Let h be some C^2-smooth Hermitian metric on X. Then there exists an Hermitian metric h_1 and a function $c : \mathbf{R}^+ \longrightarrow \mathbf{R}^+$ (both of class C^2) such that:*

(i) $h_1 \geq h$;

(ii) the (q,q) - form $\Omega = dd^c[(c \circ \rho) w_{h_1}^{q-1}]$ is strictly positive on X.

Here w_h is the (1,1)-form canonically associated with h. In our case we need $X = \Delta^{n+q} \times Y$ and we shall use only the fact that on X there exists a strictly positive (q,q)-form which is dd^c-exact: in fact d-exactness is going to be sufficient for us. We recall that a (q,q)-form Ω is called strictly positive if for any $x \in X$ and linearly independent vectors $v_1, ..., v_q \in T_x X$ one has $\Omega_x(iv_1 \wedge \bar{v}_1, ..., iv_q \wedge \bar{v}_q) > 0$.

2. Proof of the Theorem.

Step 1. Case $n = 1$.

Let $f : H_1^q(r) \longrightarrow Y$ be our meromorphic mapping. Let us denote by W the biggest open subset of Δ^1 such that f extends meromorphically to $H_W(r) := (\Delta^1 \times A^q(1-r,1)) \cup$

$(W \times \Delta^q)$, and let us remark explicitly that the complex space $X = \Delta^{1+q} \times Y$ is (obviously) q-complete.

We apply Barlet's Theorem, see §1, by taking as ρ a strictly q-convex exhaustion of X in order to have a strictly positive dd^c-exact (q,q)-form Ω on X. Let w be a fixed $(q-1,q-1)$-form of class C^2 such that $dd^c w = \Omega$. Let us denote by F the set of points of indeterminancy of f.

By shrinking the polydisc Δ^{1+q}, we can suppose, withought loss of generality, that f_z is defined in the neighborhood of $\bar{\Delta}^q$ for all $z \in W$. In the same way we can suppose that $w \in C^2(\bar{\Delta}^{1+q} \times Y)$, i.e. is smooth up to the boundary.

We need to prove that $W = \Delta$. Suppose not, and fix a point $z_0 \in \partial W \cap \Delta$. Denote by V some disc centered at z_0 which whichis contained in Δ. For $z \in V \cap W$ one has

$$vol(\Gamma_{f_z}) = \int_{\Gamma_{f_z}} \Omega = \int_{\Gamma_{f|_{\partial \Delta_z^q}}} d^c w \leq C \tag{6}$$

where the constant C does not depend on $z \in V \cap W$, while $d^c w$ iso f class C^1 on $\bar{\Delta}^{1+q} \times Y$.

To obtain estimate (1) we had used the fact that we can measure the volumes of analytic sets of pure dimension q contained in some compact part of X by means of $\int \Omega$ with Ω a strictly positive (q,q)-form on X.

We are going to check if the conditions of the Continuity Principle, mentioned in §1 are satisfied. The inequality (6) says that the second assumption of C.P. is satisfied. To check if the first one is satisfied, let us suppose that there exists a sequence $\{z_n\} \subset V \cap \Omega$, converging to $z_\infty \in \Delta$, such that $\{\Gamma_\nu := \Gamma_{f_{z_\nu}}\}$ is not contained in any relatively compact subset of $\bar{\Delta}^{1+q} \times Y$. If ν is big enough, the restriction $\rho|_{\Gamma_\nu}$ will have then a strict maximum in the interior of Γ_ν. This is impossible because the Levi form of $\rho|_{\Gamma_\nu}$ has at least one positive eigenvalue at each point of Γ_ν. Remark also that q-complete space Y cannot containe any compact q-dimensional subspace. C.P. says now (since W is not contained in any proper analytic subset of any neighborhood of z_0) that f meromorphically extends to $V_1 \times \Delta^q$ for some neighborhood V_1 of z_0 in Δ. This proves that $W = \Delta$.

Step 2. Case $n \geq 2$.

This will done by induction on n. As is follows from Lemma 1, all we need is to extend the mappings from $E_n^q(r)$ to Δ^{n+q}. For $n = 1, E_1^q(r) = H_1^q(r)$ and thus this is already done by Step 1.

Notice that $E_{n+1}^q(r) = \Delta \times E_n^q(r)$. Denote $E_{n,z}^q(r) := \{z\} \times E_n^q(r)$ for $z \in \Delta$. Remark that by the induction hyposethis the restriciton $f|_{E_{n,z}^q(r)}$ meromorphically extends onto $\Delta_z^{n+q} := \{z\} \times \Delta^{n+q}$ for all $z \in \Delta$. We denote by W the maximal open subset in Δ such that our map f extends meromorphically onto $W \times \Delta^{n+q}$.

Put $S = \Delta \setminus W$ and consider a family $\{\Gamma_{f_s} : s \in S\}$ of analytic subsets in $X := \Delta^{n+q} \times Y$. Here, as usually by Γ_{f_s} we denote the graph of the restriction $f_s := f|_{\Delta_s^{n+q}}$. Define $S_k := \{s \in S : vol(\Gamma_{f_s}) \leq k \cdot \frac{\nu}{2}\}$. Where ν is from Lemma 2 with $W = \Delta^{n+q}, W_0 = \Delta_{1-r/2}^{n+q}$. By maximality of S and by Lemma 2 we see that all points of each S_k are locally regular, thus each S_k is polar. So S is a polar subset of Δ, in other words it is a set of harmonic measure zero in Δ.

By some linear coordinate transformation in $\mathbf{C^{1+n+q}}$ we are going to change a little bit the band of the Δ^{n+q}-direction, in order to prove in the same manner that f meromorphically extends to the whole of Δ^{1+n+q}. In fact, let us consider linear changes L of the coordinate system in $\mathbf{C^{1+n+q}}$ whose associated matrices are of the form (L_1, L_2), where L_1 is (a number) close to zero and L_2 is close to the identity map of $\mathbb{C}^{n+q}$ into itself. For each L of this form we can extend f onto $\Delta^{1+n+q} \setminus L^{-1}(S^L \times \Delta^{n+q})$, where S^L is a set of harmonic measure zero in Δ. Σ in appropriate coordinate system is a $(1+n+q)$-product of the closed sets of harmonic measure zero on the plain. Thus Σ is pluripolar and of Haussdorff dimension zero. Using the fact that $\Delta^{1+n+q} \times Y$ is (obviously!) $(n+q)$ - complete and Lemma 3 we can remove the singularity Σ.

q.e.d.

3. Consequences and open questions.

Let us start with some direct consequences of the Theorem.

Corollary 1. (Thullen type extension Theorem) *Let $\Omega \subset \mathbf{C^n}$ be an open subset, $V \subset \Omega$ be an analytic subvariety of dimension q and G be an open subset of Ω which intersects every q-dimensional branch of V. Every meromorphic mapping $f : (\Omega \setminus V) \cup G \longrightarrow Y$, where Y is a q-complete complex space, extends to a meromorphic mapping from Ω to Y.*

In particular one has

Corollary 2. (Riemann type extension Theorem) *Let $\Omega \subset \mathbf{C^n}$ be an open subset, $V \subset \Omega$ be an analytic subvariety of dimension $q-1$. Every meromorphic mapping $f : \Omega \setminus V \longrightarrow Y$, where Y is a q-complete complex space, extends to a meromorphic mapping from Ω to Y.*

The proofs of Corollaries 1 and 2 are immediate after [S_1], p.5.

A general Thullen type extension Theorem for meromorphic mappings is proved by Siu when Y is compact Kähler in [S_2]. We have also:

Corollary 3. *Let Y be a complex analytic space of dimension q and let us suppose that every irreducible component of Y of dimension q is non compact. Every meromorphic mapping $f : H_n^q(r) \longrightarrow Y$, extends to a meromorphic mapping from Δ^{n+q} to Y.*

In fact, every complex space of dimension n with no compact irreducible component of dimension n is n-complete, by a Theorem of Ohsawa, [O] Th. 1.

We shall end with discussing some open questions, which naturally arise from the results and attempts of this paper.

Question 1. Let Y be a compact complex three-fold. Prove that every meromorphic (or holomorphic) map $f : H_1^2(r) \to Y$ extends onto $\Delta^3 \setminus \{$discrete set of points$\}$.

In particular, if $K \subset\subset \Delta^3$ with connected complement then every meromorphic map $f\Delta^3 \setminus K \to Y$ extends onto Δ^3 minus finite set of points.

For the proof of such type of statements one can try to use special metrics on Y. Namely a compact complex three-fold possesses a Hermitian metric h, such that its associated $(1,1)$-form ω_h satisfies $dd^c\omega_h^2 = 0$. This can help to bound the volumes of the images of two-disks in Y.

The next question comes out when one tries to prove the Corollary from the Introduction withough assuming Y to be noncompact.

Question 2. Let Y is a compact complex manifold (space) of dimension $q \geq 2$. Suppose that there exists a meromorphic map $f : \mathbf{B}_*^{\mathbf{q+1}} \to \mathrm{Y}$ from punctured ball in $\mathbb{C}^{q+1}$ onto Y such that for any $\varepsilon > 0$ the restriction $f_\varepsilon := f\mid_{\mathbf{B}_*^{\mathbf{q+1}}(\varepsilon)}$ of f onto the punctured ε-ball is still surjective. Prove that Y is Moishezon.

In the case of positive answer to this question, one can extend this f meromorphically to zero.

In fact one need somewhat stronger stetment. Let $M \ni 0$ be a strongly pseudoconvex hypersurface in the ball $\mathbf{B}_*^{\mathbf{q+1}}$, which divides it into two parts B^+ and B^-. Let a meromorphic map $f : B^+ \to Y$ as in Question 2 is given. Suppose that M is concave from the side of B^+.

Prove that if for any $\varepsilon > 0$ the restriction $f\mid_{\mathbf{B}_\varepsilon^{\mathbf{q+1}}\cap\mathbf{B}^+} : \mathbf{B}_\varepsilon^{\mathbf{q+1}} \cap \mathbf{B}^+ \to \mathbf{Y}$ is surjective, then f extends meromorphically to zero.

Question 3. Can one remove the condition $n=1$ from Lemma 3?

The attepts leed to the "non analytic" version of Remmert proper mapping theorem and to the questions of locall flattenings. The major problem here is that $\mathcal{C}_{f,C}$ will be not an analytic space now and F will be not proper in general.

R E F E R E N C E S

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